Geometry of moduli spaces of curves of genus 0 and multiple zeta values

Complex analytic description

Definition

The *moduli space of curves of genus* 0 *with n marked points* $\mathcal{M}_{0,n}$ is the set of Riemann spheres with n marked points modulo isomorphisms of Riemann surfaces (analytic structure) sending marked points to marked points.

Remark

In the genus g case, the definition is the same but for *Riemann sphere* which is replaced by *Riemann surfaces of genus* g.

We can see that the moduli space of curves of genus 0 with $n+3$ marked points is isomorphic to

$$\mathcal{M}_{0,n+3} = \{(z_0, \ldots, z_{n+2}) \in \mathbb{P}^1(\mathbb{C}) \text{ such that } z_i \neq z_j\} / \mathrm{PSL}_2(\mathbb{C}).$$

Examples

As $\mathrm{PSL}_2(\mathbb{C})$ is three transitive we can choose as representatives (modulo the action of $\mathrm{PSL}_2(\mathbb{C})$) the tuples $(0, t_1, t_2, \dots, t_n, 1, \infty)$ setting

$$t_i = \frac{z_i - z_0}{z_i - z_{n+2}} \frac{z_{n+1} - z_{n+2}}{z_{n+1} - z_0}.$$

This lead to the following identifications:

Example

- When $n = 1$ we have: $\mathcal{M}_{0,4} \simeq \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.
- When $n = 2$ we have:

$$\mathcal{M}_{0,5} \simeq (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})^2 \setminus \{t_1 \neq t_2\}.$$

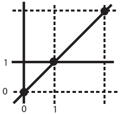


Figure: $\mathcal{M}_{0,5}$ in $\mathbb{P}^1(\mathbb{C})^2$

Remark

The previous identifications are depending on the choice of the cross-ratio, however

$$\mathcal{M}_{0,n+3} \simeq (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})^n \setminus \{\text{fat diagonal}\}.$$

Metric description

A Riemann sphere with n ($n \geqslant 3$) removed points is an hyperbolic surface.

$\mathcal{M}_{0,n}$ can be seen as all the possible hyperbolic metrics on that Riemann sphere without n points modulo isomorphisms (isometries respecting the marked points).

Definition

A pant cut of an hyperbolic surface (genus 0) is the data of $n-3$ simple loops (that do not intersect) such that cutting along the loop leads to have pants.

The length of the loop of a pant cut is a geodesic of the metric and therefor an important element to characterize it.

Compactification

The open space $\mathcal{M}_{0,n}$ can be compactified in a meaningful way. Let $\overline{\mathcal{M}_{0,n}}$ denote this compactification. The space $\overline{\mathcal{M}_{0,n}}$ classifies the stable curves of genus 0.

Analytic point of view :

- In the genus 0 case, a point in a codimension 1 component of $\partial\overline{\mathcal{M}_{0,n}}$ is two spheres glue together, the n marked points being spread on the two spheres (the double points excluded) in such a way that, there are at least 2 marked points on each sphere.
- A point in a codimension k component will be k spheres glued together the marked points being spread on the sphere.
- The gluing points together with the marked one are called special points. The marked points are spread on the k sphere such that each sphere have at least 3 special points.

Compactification

Metric-codimension 1 stratum

When a point is moving toward the boundary of $\mathcal{M}_{0,n}$ the length of one of the loop of the "pant cut" tends to 0.
The stratum is uniquely determined by the choice of that loop.
A *codimension k component* is defined by the vanishing of the length of k loops of a "pant cut".

Algebraic description

Proposition (P. Deligne and D. Mumford ([DM69]))

The space $\overline{\mathcal{M}_{0,n}}$ is scheme over $\mathbb{Z}$. It is irreducible and its boundary is a normal crossing divisor.

Combinatorial description of the boundary of $\overline{\mathcal{M}_{0,n}}$

Stratification

A description of the boundary of $\overline{\mathcal{M}_{0,n}}$ is:

- The irreducible component of $\partial\overline{\mathcal{M}_{0,n}}$ is the product of some $\overline{\mathcal{M}_{0,k}}$ for $k \leqslant n$.
- Components of codimension 1 are of the the type $\overline{\mathcal{M}_{0,k}} \times \overline{\mathcal{M}_{0,n-k-1}}$.
- A codimension k component is the intersection of k component of codimension 1.

Espace de modules de courbes, groupes modulaires et thÃ©orie des champs, Panorama et SynthÃ¨se, no. 7, SMF, 1999.

Strata of codimension 1

- Moving in a codimensin 1 stratum makes move the marked points but they stay on the same sphere.
  We have then an unordered partition $\sigma_1|\sigma_2$ of the marked points $\{z_1, \ldots, z_n\}$.
- In *the metric description of $\overline{\mathcal{M}_{0,n}}$:* the stratum is determined by a loop around the points in σ_1 (or σ_2).
- Each codimension 1 stratum is uniquely determined by the corresponding unordered partition.
  We represent each of these strata by a stable partition $\sigma_1|\sigma_2$ of $\{z_1, \ldots, z_n\}$.
- For example in $\overline{\mathcal{M}_{0,4}}$ the partition $z_1z_3|z_2z_4$ corresponds to the stratum defined by the vanishing of the length of the loop around the points z_1 and z_3.

Example

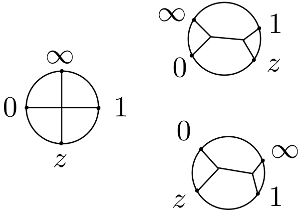


Figure:

Codimension 1 strata for $n = 5$

- A codimension 1 stratum is given by a loop around 2 points (a loop around 3 is the same as one around 2).
- There are $\binom{5}{2} = 10$ codimension 1 strata.

strata	$1\infty\|0z_1z_2$	$0\infty\|z_1z_21$	$01\|z_1z_2\infty$	$0z_1\|z_21\infty$	$0z_2\|z_11\infty$
strata	$1z_1\|z_20\infty$	$1z_2\|\infty 0z_1$	$\infty z_1\|z_210$	$\infty z_2\|z_110$	$z_1z_2\|01\infty$

- $\mathcal{M}_{0,5} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus$ seven lines.
- $\overline{\mathcal{M}_{0,5}} = (\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{\text{seven lines }\}) \bigcup \{\text{ten lines}\}$.

Codimension 1 strata for $n = 6$

A loop can be around :

- 2 points (or 4 looking at the complement) and then $\binom{6}{2} = 15$ stratum,
- or 3 points (other 3 other ...) so $\binom{6}{3} \cdot 1/2 = 10$ other strata.
- There are 25 strata.

Tree of projective lines

- Points on a codimension k stratum are $k+1$ copy of $\mathbb{P}^1$ that intersect on the double points.
- The marked points are on the $k+1$ $\mathbb{P}^1$ such that each $\mathbb{P}^1$ have at least 3 special points.
- The marked points stay in the same $\mathbb{P}^1$ as one move in the stratum.
- A stratum is then uniquely determined by a tree of projective lines (intersection are the double points) together with n marked points on the edge.

Some examples :

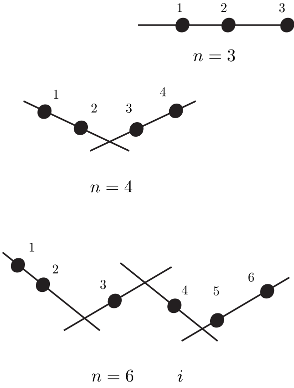
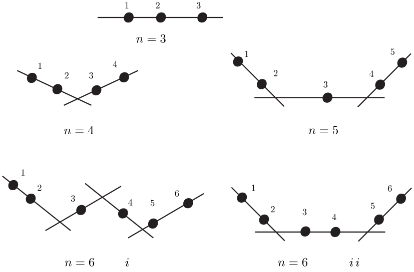


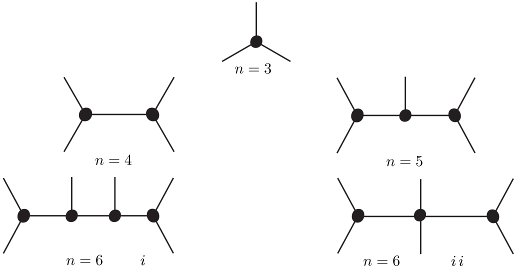
Figure: Except for the case $n = 6$ (ii(, we have represented only maximal $(n-3)$ codimension stratum (points).The case $n = 6$ (ii) is of codimension 2.

Planar trees

This representation is dual to the former :

- Special points are edges, double points being internal edges and marked points being external one. Sphere (or the $\mathbb{P}^1$) are vertices.
- Two edges share a vertices if and only if the corresponding points are on the same sphere.


1 2 3
$n = 3$
1 2 3 4
$n = 4$
1 2 3 4 5
$n = 5$
1 2 3 4 5 6
$n = 6$ i
1 2 3 4 5 6
$n = 6$ ii


$n = 3$
$n = 4$
$n = 5$
$n = 6$ i
$n = 6$ ii

The example of $\zeta(2)$

As seen in the introduction, $\zeta(2)$ can be seen as an integral on $\overline{\mathcal{M}_{0,5}}$.

Differential form

The fact that $\mathcal{M}_{0,5} \simeq \{(t_1, t_2) \in (\mathbb{P}^1 \setminus \{0,1,\infty\}) \times (\mathbb{P}^1 \setminus \{0,1,\infty\}) \,|\, t_1 \neq t_2\}$ gives us two coordinates on $\mathcal{M}_{0,5}$ that are t_1 and t_2. We then can define a meromorphic differential form on $\overline{\mathcal{M}_{0,5}}$

$$\omega_2 = \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2}.$$

Integration domain

The identification of $\mathcal{M}_{0,5}$ with $(\mathbb{P}^1 \setminus \{0,1,\infty\})^2$ allows us to lift the 2 simplex $\{0 < t_1 < t_2 < 1\}$ in $\mathcal{M}_{0,5}$ and to look at its "algebraic" boundary. We will write Φ_5 for that *simplex* in $\mathcal{M}_{0,5}$.

Boundary of $\overline{\mathcal{M}_{0,5}}$:

- $\mathcal{M}_{0,5} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus$ seven lines :

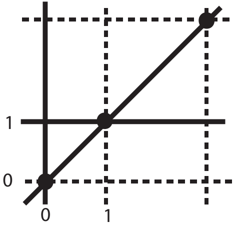


Figure:

- $\partial\overline{\mathcal{M}_{0,5}}$ is ten lines : the seven and 3 others that are the exceptional divisors of the blow up at $(0,0)$, $(1,1)$, (∞,∞).
- $\overline{\mathcal{M}_{0,5}} = \left(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{7 \text{ lines}\}\right) \bigcup \{10 \text{ lines}\}$.

Divisors of singularities

Let A be the divisor of the singularities of the differential form ω_2

- The divisor A is not the whole preimage of the singularities in $\mathbb{P}^1 \times \mathbb{P}^1$
- The exceptional divisors at $(0,0)$ and $(1,1)$ are not component of A.
- Stratum of the boundary of $\overline{\mathcal{M}_{0,5}}$ are divided in two categories:
- 5 components are the divisor A
- 5 other are the boundary B of Φ_5.

divisor A of singularities ω	$0z_2\|z_1 1\infty$	$1z_1\|z_2 0\infty$	$\infty z_1\|z_2 10$	$\infty z_2\|z_1 10$
				$\mathbf{01\|z_1 z_2 \infty}$
boundary B	$0z_1\|z_2 1\infty$	$z_1 z_2\|01\infty$	$1z_2\|\infty 0 z_1$	$\mathbf{1\infty\|0 z_1 z_2}$
				$\mathbf{0\infty\|z_1 z_2 1}$

Some pictures

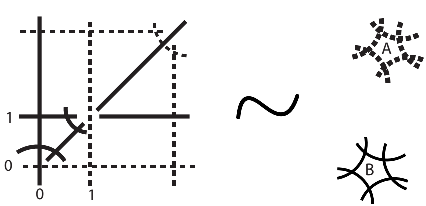


Figure:

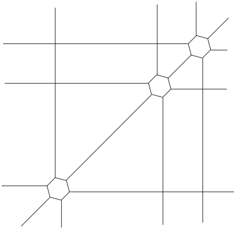


Figure: Real points of $\overline{\mathcal{M}_{0,5}}$

In this example appears the question of controlling how singularities behave in respect with blow up.

Forgetful maps

Let $n \geqslant 3$ and S a finite ordered set with $|S| = n$. We write $\overline{\mathcal{M}_{0,S}}$ for $\overline{\mathcal{M}_{0,|S|}}$. Let S' be a sub ordere set of S with $|S'| \geqslant 3$. Then we have a canonical morphism, forgetful map (with $T = S \setminus S'$)

$$\phi_T : \overline{\mathcal{M}_{0,S}} \rightarrow \overline{\mathcal{M}_{0,S'}}$$

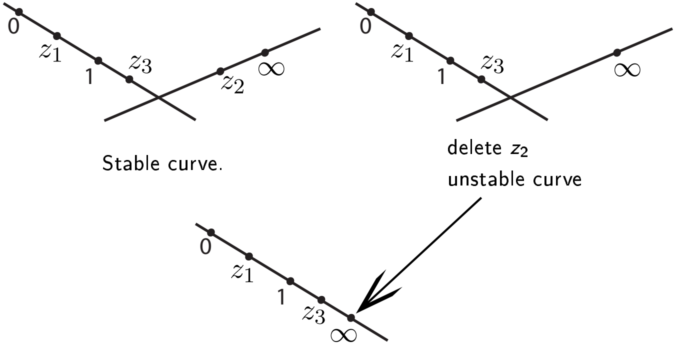
which delete the point indexed by elements of T and "smooth" the unstable component.

Example with $\mathcal{M}_{0,5}$

- The case $T = \{z_2\}$: the stratum $0z_1z_2|1\infty$ is map to $0z_1|1\infty$
- The case $T = \{z_2\}$: the stratum $0z_2|z_11\infty$ is map to $\overline{\mathcal{M}_{0,4}}$

Some pictures

In $\overline{\mathcal{M}_{0,6}}$, $S = \{0, z_1, z_2, z_3, 1, \infty\}$, $T = \{z_2\}$ lets have a look to the component defined by $z_2\infty|0z_1z_31$:



Stable curve.

delete z_2
unstable curve

The "smoothing" or "contracting" is done in putting the last label at the node place

Coordinates, functions and differential forms

- Choose a cross ratio on $\overline{\mathcal{M}_{0,4}}$: $\frac{\tilde{z_1}-\tilde{z_0}}{\tilde{z_1}-\tilde{z_4}}\frac{\tilde{z_3}-\tilde{z_4}}{\tilde{z_3}-\tilde{z_0}}$.
- That is the same as identifying $\mathcal{M}_{0,4}$ with $\{z \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ and the stratum of $\overline{\mathcal{M}_{0,4}}$, $\tilde{z_0}\tilde{z_1}|\tilde{z_2}\tilde{z_4}$, $\tilde{z_2}\tilde{z_1}|\tilde{z_0}\tilde{z_4}$, $\tilde{z_4}\tilde{z_1}|\tilde{z_0}\tilde{z_4}$ with respectively $z=0,1$ and ∞.
- For $\overline{\mathcal{M}_{0,n}}$ we choose a system of representative:

$$\mathcal{M}_{0,n+3} \simeq \{(0,z_1,\ldots,z_n,1,\infty)\,|z_i \neq z_j \text{ for } i\neq j \text{ and } \forall i\, z_i \neq 0,1,\infty\}.$$

- We have coordinate functions t_i such that $t_i(0,z_1,\ldots,z_n,1,\infty)=z_i$. They are the pull back of the standard affine coordinates on $\mathbb{P}^1=\overline{\mathcal{M}_{0,4}}$ by the forgetful map $\phi^S=\phi_T$ with $S=\{0,1,\infty,z_j\,/\,j\neq i\}$.
- We will write z_i for this i-th coordinates (sometimes).

General situation

Choose two subsets S and S' of $S_0 = \{\tilde{z_0}, \ldots, \tilde{z_{n+2}}\}$ such that $|S \cap S'| = 3$ and $S_0 = S \cup S'$. Then we have a product of forgetful map $\phi^S \times \phi^{S'}$

$$\overline{\mathcal{M}_{0,S_0}} \longrightarrow \overline{\mathcal{M}_{0,S}} \times \overline{\mathcal{M}_{0,S'}}$$

which is an isomorphism on the open spaces.
Let C be a codimension 1 stratum of $\overline{\mathcal{M}_{0,S_0}}$.

- If C is stable under both ϕ^S and $\phi^{S'}$ then it is crashed done.
- If C is stable under only one map, then usually the image of C is still a codimension one stratum in the product.

The projection $\overline{\mathcal{M}_{0,n+3}} \rightarrow (\mathbb{P}^1)^n$

The projection $p : \overline{\mathcal{M}_{0,n+3}} \rightarrow (\mathbb{P}^1)^n$ is an extension of the natural projection $\mathcal{M}_{0,n+3} \rightarrow (\mathbb{P}^1)^n$ which send $(0, z_1, ..., z_n, 1\infty)$ to $(z_1, ..., z_n)$.

Question

In the case $n = 3$ what is the image of the component given by $0z_1z_3|z_21\infty$?

- A geodesic surrounding $0, z_1$ and z_3 have a length that tends to 0 when it tends to the boundary.
- Symbolically we have $0 = z_1 = z_3$ which is the equation of a line in $(\mathbb{P}^1)^3$. The component $0z_1z_3|z_21$ maps to that line ...

Description

In order to obtain a description of the image of the boundary component, we say that the points in the same subset of the partition are equals. More precisely

- Components of types $s_i s_j|...$, $s_i\varepsilon|...$ with $\varepsilon \in \{0, 1, \infty\}$, give hyperplanes $x_i = x_j$ and $x_i = 0, 1, \infty$;
- Partition of types $\{3 \text{ points}\}|...$ (with at most one being $0, 1, \infty$) give codimension 2 affine space ;
- ...
- Partitions of types $\varepsilon z_1 \ldots z_n|ab$ (with $\varepsilon = 0, 1, \infty$) give the points $(0, ..., 0)$ $(1, ..., 1)$ and $(\infty, ..., \infty)$.

Forgetful maps and $(\mathbb{P}^1)^n$

- The projection $\overline{\mathcal{M}_{0,S}} \longrightarrow (\mathbb{P}^1)^n$ is the product of forgetful maps $\phi^{S_1} \times \cdots \phi^{S_n}$ with $S_i = \{\tilde{z_0}, \tilde{z_i}, \tilde{z_{n+1}}, \tilde{z_{n+2}}\}$.
- It is equivalent to the composition of maps

$$\underbrace{\overline{\mathcal{M}_{0,n+3}} \longrightarrow \overline{\mathcal{M}_{0,n+2}} \times \overline{\mathcal{M}_{0,4}}}_{f_n} \longrightarrow \overline{\mathcal{M}_{0,n+1}} \times \overline{\mathcal{M}_{0,4}} \times \overline{\mathcal{M}_{0,4}}$$

$$\longrightarrow \cdots \longrightarrow (\overline{\mathcal{M}_{0,4}})^n$$

- The image of the component $z_n\sigma_1|\sigma_2 01\infty$ ($\sigma_1 \cup \sigma_2 = \{z_1, \ldots, z_{n-1}\}$) is crashed down (even if it is unstable on the second factor). It is a sort of diagonal.

Example in $\overline{\mathcal{M}_{0,5}}$

- $\mathbf{01|z_1z_2\infty}$
- $\mathbf{1\infty|0z_1z_2}$
- $\mathbf{0\infty|z_1z_21}$

example in $\overline{\mathcal{M}_{0,6}}$

- $1\infty|0z_1z_2z_3 \mapsto$ point
- $z_1z_2z_3|01\infty \mapsto$ line
- $01z_3|z_1z_2\infty \mapsto$ line

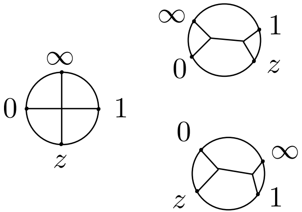
Real points of $\overline{\mathcal{M}_{0,n+3}}(\mathbb{R})$ ([GM02][prop.2.1])

- It is a connected closed real manifold.
- Stratification leads to a cell decomposition.
- Cells of it are in one-to-one correspondence with stable locally planar $(n+3)$-labeled trees.
- The relation "a cell is a codimension one component of the boundary of another cell" corresponds to the relation "a locally planar tree produces another locally planar tree by contracting an internal edge."
- Any open cell is determined by an unoriented cyclic order on $\{0, \ldots, n+2\}$.
- Once the order fixed, the choice of 3 points allows us to identify the open cell with the simplex Δ_n (via real coordinates).
- The closure of each open cell has the structure of a Stasheff polytope.
- Strata of codimension 1 of a cell are indexed by those stable 2-partitions of S which are compatible with the respective cyclic order.

Some comments

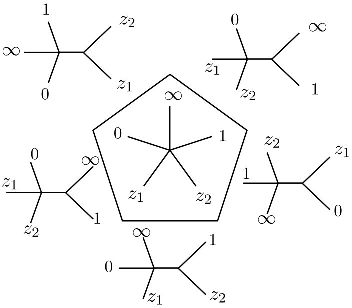
- As said, tending to the boundary is the same as the length of a geodesic tending to 0.
- This geodesic intersects the equator in two points.
- At the limit the equator has became two equators.
- Staying in $\overline{\mathcal{M}_{0,n+3}}(\mathbb{R})$, the marked points are on the real equator and at the limit, the partition is given by cutting the equator in two.
- The partition keeps the order of the cell we were in.

Example



$n = 1$. Boundary of the standard cell defined by $0 < z < 1 < \infty$.

Example



$n = 2$. Boundary of the standard cell defined by $0 < z_1 < z_2 < 1 < \infty$

Standard cell

We call *standard cell* Φ_n, the real open cell of $\mathcal{M}_{0,n+3}(\mathbb{R})$ corresponding to the cyclic order

$$0 < z_1 < \ldots < z_n < 1 < \infty.$$

It is the preimage of $\Delta_n = \{0 < t_1 < \ldots < t_n < 1\} \subset \mathbb{P}^1(\mathbb{R})^n$ induced by the map

$$\begin{array}{rcl} \mathcal{M}_{0,n+3} & \longrightarrow & (\mathbb{P}^1)^n \\ (0, z_1, \ldots, z_n, 1, \infty) & \longmapsto & (z_1, \ldots z_n). \end{array}$$

Differential forms associated and MZV

Let $\mathbf{k} = (k_1, \ldots, k_p)$ be a p-tuple of integer ($k_1 \geqslant 2$ and $k_1 + \ldots + k_p = n$).
We associate to $\mathbf{k}$ the n-tuple

$$\varepsilon_{\mathbf{k}} = (\varepsilon_n, \ldots, \varepsilon_1) = (\underbrace{0\ldots,0}_{k_1 \text{ times}}, 1, \ldots, \underbrace{0\ldots,0}_{k_n \text{ times}}, 1)$$

and the differential form in $\Omega_{log}(\overline{\mathcal{M}_{0,n+3}})$

$$\omega_{\mathbf{k}} = \frac{dz_1}{z_1 - \varepsilon_1} \wedge \ldots \wedge \frac{dz_n}{z_n - \varepsilon_n}.$$

Distinguished 2 partitions

Let ε be an n-tuple of 0 and 1.

Definition

1. Let $\alpha \in \{0, 1, \infty\}$ we define $S(\alpha, \varepsilon)$ by:

$$S(0, \varepsilon) = \{z_i \text{ with } i \text{ such that } \varepsilon_i = 0\}$$

$$S(1, \varepsilon) = \{z_i \text{ with } i \text{ such that } \ \varepsilon_i = 1\} \qquad S(\infty, \varepsilon) = S(0, \varepsilon) \cup S(1, \varepsilon.)$$

2. A 2 partition of $\{0, z_1, \ldots, z_n, 1, \infty\}$ is of type α respecting ε if it is of the form

$$\alpha T|... \qquad \text{with } T \subset S(\alpha, \varepsilon).$$

Main result

Proposition

The divisor of singularities of $\omega_{\mathbf{k}}$ in $\overline{\mathcal{M}_{0,n+3}}$ is the union $A_{\mathbf{k}}$ of the divisor corresponding to the stable 2-partition of some type α respecting $\varepsilon_{\mathbf{k}}$.

Corollary

The divisor $A_{\mathbf{k}}$ does not intersect the boundary of Φ_n in $\overline{\mathcal{M}_{0,n+3}}(\mathbb{R})$.
We have the following equality

$$\int_{\Phi_n} \omega_{\mathbf{k}} = \zeta(k_1, \ldots, k_p).$$

Two strategies and a key lemma

- By induction looking maps at $\overline{\mathcal{M}_{0,n+3}} \longrightarrow \overline{\mathcal{M}_{0,n+2}} \times \overline{\mathcal{M}_{0,4}}$ and the Keel description of those maps.
- Looking at the projection $\overline{\mathcal{M}_{0,n+3}} \longrightarrow (\mathbb{P}^1)^n$.

Lemma ([Gon02][lemma 3.8])

Let Y be a normal crossing divisor in a smooth variety X and $\omega \in \Omega^n_{log}(X \setminus Y)$. Let $p : \widehat{X} \longrightarrow X$ be the blow up of an irreducible variety Z. Suppose that the generic point of Z is different from the generic points of strata of Y. Then $p^*\omega$ does not have a singularity at the special divisor of $\widehat{X}$.

Further developments

1. Motivic multiple zeta values. If B_n is the Zariski closure of the boundary of Φ_n, the multiple zeta values $\zeta(k_1, \ldots, k_p)$ is a period of the motive :

$$\mathrm{H}^n(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B_n \setminus (A_{\mathbf{k}} \cap B_n)).$$

2. F. Brown have shown that all the periods of $\mathcal{M}_{O,n+3}$ are rational linear combination of MZV.
3. Q. Wang gives a similar expression of the multiple polylogarithms $Li_{k_1,\ldots,k_p}(z_1, \ldots, z_n)$ on $\overline{\mathcal{M}_{0,n+3}}$.

X. Buf, J. Fehrenbach, P. Lochak, and L. Schneps, *Espace de modules de courbes, groupes modulaires et théorie des champs*, Panorama et Synthèse, no. 7, SMF, 1999.

Pierre Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Pub. Math. Institut des Hautes Etudes Scientifiques (1969), no. 36.

A. B. Goncharov and Yu. I Manin, *Multiple ζ-motives and moduli spaces* $\overline{\mathcal{M}_{0,n}}$, e-print, www.arxiv.org/abs/math.AG/0204102, April 2002.

A. B. Goncharov, *Period and mixed motives*, e-print, www.arxiv.org/abs/math.AG/0202154, May 2002.