Geometry of moduli spaces of curves of genus 0 and multiple zeta values

Motivations.

I claim that the moduli space of curves of genus 0 with n marked points $\mathcal{M}_{0,n}$ is isomorphic to

$$\mathcal{M}_{0,n} = \{(z_0, ... z_{n+2}) \in \mathbb{P}^1(\mathbb{C}) \text{ tel que } z_i \neq z_j\} / \mathrm{PSL}_2(\mathbb{C})$$

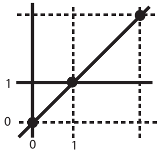
Example $(n = 5)$

As $\mathrm{PSL}_2(\mathbb{C})$ is three transitive we can chose as representatives the tuple $(0, t_1, t_2, 1, \infty)$ setting (modulo the action of $\mathrm{PSL}_2(\mathbb{C})$) $z_0 = 0$, $\frac{z_1 - z_0}{z_1 - z_4} \frac{z_3 - z_4}{z_3 - z_0} = t_1$, $\frac{z_1 - z_0}{z_1 - z_4} \frac{z_3 - z_4}{z_3 - z_0} = t_2$, $z_3 = 1$ and $z_4 = \infty$. Then we have

$$\mathcal{M}_{0,5} \simeq (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1\})^2 \setminus \{t_1 = t_2\}.$$

and some familiar picture:

Picture



Motivations

The integral representation of $\zeta(2)$ is given by the formula

$$\zeta(2) = \int_0^1 \frac{1}{t_2}\left(\int_0^{t_2} \frac{dt_1}{1-t_1}\right) dt_2 = \int\int_{0<t_1<t_2<1} \frac{dt_2}{t_2} \wedge \frac{dt_1}{1-t_1}$$

Thinking of (t_1, t_2) as affine coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, the singularities of the differential form are: $t_1 = 1$, $t_1 = \infty$, $t_2 = 0$, $t_2 = \infty$

Dessin

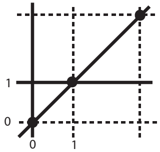


Figure:

Only points $(0;0)$, and $(1;1)$ are and issue for the integration.

Motivations

1. We have seen that:
   - the variables change $u_1 = t_2,\ u_1 u_2 = t_1$,show that there are not any problem in $(0,0)$.

$$\frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \rightsquigarrow \frac{du_1 \wedge du_2}{1-u_1 u_2}$$

   - A new change of variables in $(1,1)$ gives solve the problem

$$\frac{du_1 \wedge du_2}{1-u_1 u_2} \rightsquigarrow \frac{dv_1 \wedge dv_2}{v_1 v_2 + v_2 + 1} \quad v_1 \in [0,1],\ v_2 \in \mathbb{R}_+$$

2. Those two change of variables are in fact local expression of the blow up at $(0,0)$ and $(1,1)$ respectively.
3. There still a point where the divisor is not normal crossing: $(\infty; \infty)$. Blowing up also this point we found a compactification of $\mathcal{M}_{0,5}$ such that the boundary is normal crossing.

Complex analytic description

Definition

The *moduli space of curves of genus* 0 *with* n *marked points* $\mathcal{M}_{0,n}$ is the set of Riemann sphere with n marked points modulo ismophisms of Riemann surface (analytic structure).

Remark

In the genus g case, the definitions is the same but for *Riemann sphere* which is replaced by *Riemann surfaces of genus* g

We can see that the moduli space of curves of genus 0 with n marked points is isomorphic to

$$\mathcal{M}_{0,n} = \{(z_0, ...z_{n+2}) \in \mathbb{P}^1(\mathbb{C}) \text{ tel que } z_i \neq z_j\} / \mathrm{PSL}_2(\mathbb{C})$$

Example

- $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$
- $\mathcal{M}_{0,5} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^2 \setminus \{x_1 = x_2\}$

Metric description

If one think of the marked points as being remove from the Riemann sphere, one have an hyperbolic with and hyperbolic metric. One can think of $\mathcal{M}_{0,n}$ as all the possible hyperbolic metric on that Riemann sphere without n points modulo isomorphisms (isometries).

Definition

A pence cut of an hyperbolic surface (genus 0) is the data of $n-3$ simple loops (that do not intersect) such that cutting along the loop leads to have pence.

The length of the loop of a pence cut are geodesic of the metric and then an important element to characterize it.

Compactification

The open space $\mathcal{M}_{0,n}$ can be compactified in a meaningful way. Let $\overline{\mathcal{M}_{0,n}}$ denote this compactification. The space $\overline{\mathcal{M}_{0,n}}$ classifies the stable curve of genus 0.

Analytic point of view :

- A stable curve is a curve possibly singular that only have double points singularities. In the genus 0 case, a point in a codimension 1 component of $\partial\overline{\mathcal{M}_{0,n}}$ is two sphere glue together, the n marked points being spread on the two spheres (the double points excluded) in such a way there are at least 2 marked points on each sphere.
- A point in a codimension k component will be k spheres glued together (it have to stay simply connected) the marked points being spread on the sphere.
- The gluing points together with the marked one are called special points. The marked points are spread on the k sphere such that each sphere have at least 3 special points.

Compactification

Metric-codimension 1 stratum

When a point is moving toward the boundary of $\mathcal{M}_{0,5}$ the length of one the loop proving the "pence cut" tends to 0.
The strata is uniquely determinate by the choice of that loop.

A

codimension k component is defined by the vanishing of the length of k loops of a "pence cut".

Combinatoric description of the boundary of $\overline{\mathcal{M}_{0,n}}$

- Proposition (P. Deligne et D. Mumford ([?]))

  The space $\overline{\mathcal{M}_{g,n}}$ is irreducible and its boundary is a normal crossing divisor.

- Stratification

  A description of the boundary of $(\overline{\mathcal{M}_{0,n}})_n$ is:

  - The irreducible component of $\partial\overline{\mathcal{M}_{0,n}}$ to product of some $\overline{\mathcal{M}_{0,k}}$ for $k \leqslant n$.
  - Components of codimension 1 are of the the type $\overline{\mathcal{M}_{0,k}} \times \overline{\mathcal{M}_{0,n-k-1}}$
  - A codimension k component is the intersection of k component of codimension 1

A natural stratification on $\overline{\mathcal{M}_{0,n}}$ is such that the codimension k stratum are the irreducible component of codimension k minus those of codimension $k+1$.

[**?**][I.2. pages 25 à 44].

Strata of codimension 1

- We have seen that a point on a codimension 1 component was 2 spheres glued on a double points, the marked points being spread on the two sphere. Moving in the strata makes move the marked points but they stay on the same sphere. We have then a partition $\sigma_1|\sigma_2$ of the marked points $\{z_1, \dots, z_n\}$.
- The metric description of $\overline{\mathcal{M}_{0,n}}$ tell us this codimension 1 strata is determinated by choosing a loop around at least two marked points. This choice leads us again to the partition $\sigma_1|sigma_2$
- Each codimension 1 strata is uniquely determinate by the corresponding partition. We represent each of these strata by a stable partition of $\{z_1, \dots, z_n\}$ (each set has at least 2 elements)
- For example in $\overline{\mathcal{M}_{0,n}}$ the partition $Z_1 z_3|Z_2 z_4$ corresponds to the strata defined by the vanishing of the length of the loop around the points z_1 and z_3.

Example

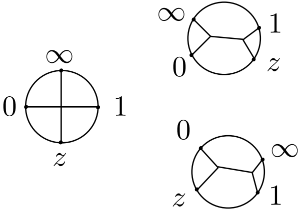


Figure:

A *codimension k component* is defined by the vanishing of the length of k loops (non intersecting) around at least two points. It is uniquely determinated by those k loops

Codimension 1 strata for $n = 5$ et $n = 6$

$n = 5$

- A codimension 1 strata is given by a loop around 2 points (a loop around 3 is the same as one around 2).
- There are $\binom{2}{5} = 10$ codimension 1 stratum.

strata	$1\infty\|0z_1z_2$	$0\infty\|z_1z_2 1$	$01\|z_1z_2\infty$	$0z_1\|z_2 1\infty$	$0z_2\|z_1 1\infty$
strata	$1z_1\|z_2 0\infty$	$1z_2\|\infty 0 z_1$	$\infty z_1\|z_2 10$	$\infty z_2\|z_1 10$	$z_1z_2\|01\infty$

- $\mathcal{M}_{0,5} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus$ seven lines
- $\overline{\mathcal{M}_{0,5}} = (\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{\text{seven lines }\}) \bigcup \{\text{ten lines}\}$

$n = 6$

A loop can be around :

- 2 points (or 4 looking at the complement)and then $\binom{2}{6} = 15$ stratum
- or 3 points (other 3 other ...) so $\binom{3}{6} * 1/2 = 10$ other

Tree of projective line

- Points on a codimension k strata are $k+1$ copy of $\mathbb{P}^1$ ($\mathbb{P}^1\mathbb{C}$) that intersect on the double points.
- The marked points are on the $k+1$ $\mathbb{P}^1$ such that each $\mathbb{P}^1$ have at least 3 special points.
- The marked points stay in the same $\mathbb{P}^1$ as one move in the strata.
- A strata is then uniquely determinated by a tree of projective line (intersection are the double points) together with n marked points on the edge.

Some examples :

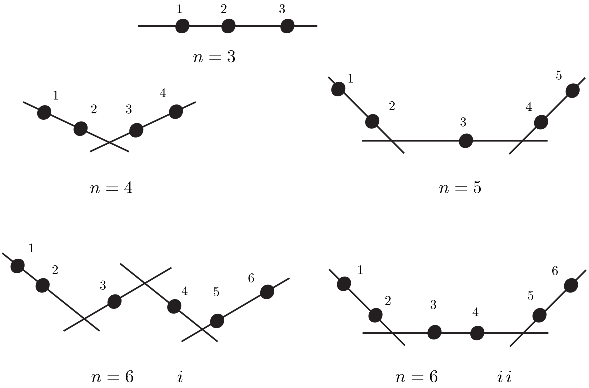


Figure: Except for the case $n = 6$ *ii*, we have represented only maximal $(n-3)$ codimension stratum (points).The case $n = 6$ *ii* is of codimension 2.

Planar trees

This representation is dual to the former :

- Special points are edges, double points being internal edges and marked points being external one. Sphere (or the $\mathbb{P}^1$) are vertices.
- Two edges share a vertices if and only if the corresponding points are on the same sphere.

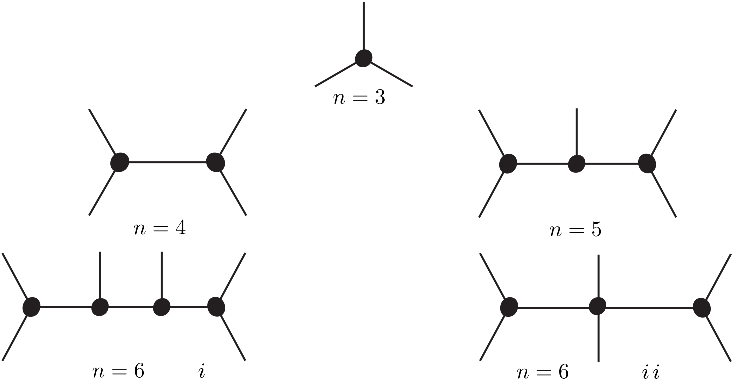


Figure:

The example of $\zeta(2)$

As seen in the introduction, $\zeta(2)$ can be seen as an integral on $\overline{\mathcal{M}_{0,n}}$.

Differential form

The fact that
$\mathcal{M}_{0,5} \simeq \{(t_1,t_2) \in (\mathbb{P}^1 \setminus \{0,1,\infty\}) \times (\mathbb{P}^1 \setminus \{0,1,\infty\}) \,|t_1 \neq t_2\}$
gives us two coordinates on $\mathcal{M}_{0,5}$ that are t_1 and t_2. We then can define a meromorphic differential form on $\overline{\mathcal{M}_{0,n}}$

$$\omega_2 = \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2}$$

Integration domain

The identification of $\mathcal{M}_{0,5}$ with $(\mathbb{P}^1 \setminus \{0,1,\infty\}$ allows us to lift the 2 simplex $\{0 < t_1 < t_2 < 1\}$ in $\mathcal{M}_{0,5}$ and to look at its "algebraic" boundary. We will write Φ_5 for that *simplex* in $\mathcal{M}_{0,5}$.

Some propositions

Proposition

Let ω be an analytic differential form on a compact variety X. Then the integral

$$\int_X \omega$$

is convergent.

Corollary

Let ω be a meromorphic differential form on a compact variety X and Φ be an open subvariety of X. Then the integral

$$\int_{\overline{\Phi}} \omega$$

is convergent if and only if the boundary of Φ is disjoint for the divisor of singularities of ω.

We need to identify the boundary of Φ_5 in $\mathcal{M}_{0,n}$ and the divisor of singularities of ω.

Boundary of $\overline{\mathcal{M}_{0_5}}$:

- $\mathcal{M}_{0,5} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus$ seven line :

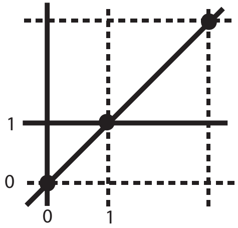


Figure:

- $\partial\overline{\mathcal{M}_{0,5}}$is ten lines : the seven and 3 others that are the exceptional divisor of the blow up at $(0,0)$, $(1,1)$, (∞,∞).
- $\overline{\mathcal{M}_{0,5}} = \left(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{7 \text{ lines}\}\right) \bigcup \{10 \text{ lines}\}$

The 10 codimension 1 stratum

Here is (again) the list of the 10 codimension 1 stratum.

stratum	$1\infty\|0z_1z_2$	$0\infty\|z_1z_2 1$	$01\|z_1z_2\infty$	$0z_1\|z_2 1\infty$	$0z_2\|z_1 1\infty$
image in $(\mathbb{P}^1)^2$	$(0,0)$	$(1,1)$	(∞,∞)	$x_1 = 0$	$x_2 = 0$
stratum	$1z_1\|z_2 0\infty$	$1z_2\|\infty 0 z_1$	$\infty z_1\|z_2 10$	$\infty z_2\|z_1 10$	$z_1z_2\|01\infty$
image in $(\mathbb{P}^1)^2$	$x_1 = 1$	$x_2 = 1$	$x_1 = \infty$	$x_2 = \infty$	$x_1 = x_2$

Divisor of singularities

Let A be the divisor of the singularities of the differential form ω_2

- The divisor A is not the whole preimage of the singularities in $\mathbb{P}^1 \times \mathbb{P}^1$
- The exceptional divisor at $(0,0)$ and $(1,1)$ are not component of A. Their are the *new lines* that appear in our previous blow up.
- Stratum of the boundary of $\overline{\mathcal{M}_{0,n}}$ are divide in two categories:
- 5 components are the divisor A
- 5 other are the boundary B of Φ_5.

divisor A of singularities ω	$0z_2\|z_1 1\infty$	$1z_1\|z_2 0\infty$	$\infty z_1\|z_2 10$	$\infty z_2\|z_1 10$
				$\mathbf{01\|z_1 z_2 \infty}$
boundary B	$0z_1\|z_2 1\infty$	$z_1 z_2\|01\infty$	$1z_2\|\infty 0 z_1$	$\mathbf{1\infty\|0 z_1 z_2}$
				$\mathbf{0\infty\|z_1 z_2 1}$

Exeptional divisors are in bold.

Un dessin

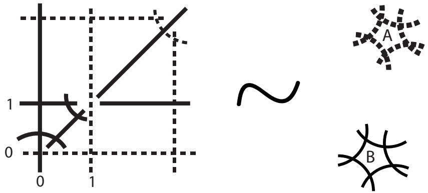
- 

Figure:

- In this example appears the question of how controlling what how singularities behave in respect with blow up

Coordinates, functions and differential forms

- Chosing 3 marked point to sent them on 0, 1 and ∞ we chose a system of representative:

$$\mathcal{M}_{0,n+3} \simeq \{(0, z_1, \ldots, z_n, 1\infty) \, | z_i \neq z_j \text{ for } i \neq j \text{ and } \forall i \; z_i \neq 0, 1, \infty\}$$

- We have coordinates function t_i such that $t_i(0, z_1, \ldots, z_n, 1, \infty) = z_i$. They are the pull back of the standard affine coordinates on $\mathbb{P}^1$ by the forgetful map.
- We will write z_i for this i-th coordinates (sometimes).

Proposition

The following proposition ([?][prop.2.1]) describe the real structure of $\overline{\mathcal{M}_{0,n}}(\mathbb{R})$.

1

The set $\overline{\mathcal{M}_{0,n}}(\mathbb{R})$ is a connected closed real manifold.Connected components of intersections of $\overline{\mathcal{M}_{0,n}}(\mathbb{R})$ with complex boundary strata form a cell decomposition. Cells of it are in one-to-one correspondence with stable locally planar $(n+3)$-labeled trees. The relation "a cell is a codimension one component of the boundary of another cell" corresponds to the relation "a locally planar tree produces another locally planar tree by contracting an internal edge."

.

Proposition

2

Fix an unoriented cyclic order on $\{1, \ldots, n+3\}$ and consider the respective open cell. Any choice of three consecutive labels with respect to this order allows one to introduce real coordinates which identify the open cell with the simplex Δ_n, this identification has been noted Φ_n

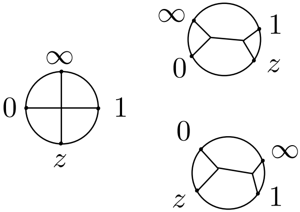
3

The closure of each open cell has the structure of a Stashe polytope. In particular, its boundary strata of codimension 1 are indexed by those stable 2-partitions of S which are compatible with the respective cyclic order: they correspond to breaking the real equator into two connected arcs.

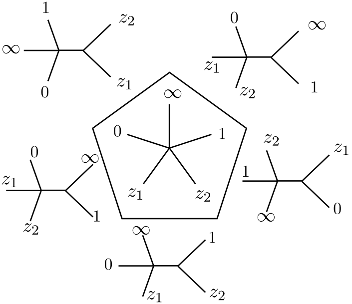
Some comments

- As said, tending to the boundary is the same as the lengh of a geodesic tending to 0
- This geodesic interstect the equator in two points.
- At the limit the equator has became two equator.
- Staying in $\overline{\mathcal{M}_{0,n}}(\mathbb{R})$, the marked points are on the real equator and at the limit, the partition is given by cutting the equator in two.
- The partition keep the order of the cell we were. partie.

Example



$n = 1$.Boundry of the standard cell is defined by $0 < z < 1 < \infty$



$n = 2$.Boundray of the sttandard cell defined by $0 < z_1 < z_2 < 1 < \infty$

Blow-up

Proposition

Let X be a variety and Z a sub-variety of X. Then there exist a unique $(\widehat{X}, \pi)$ (up to isomorphism) such that:

1. $\varepsilon : \widehat{X} \to X$
2. $\widehat{X} \setminus \pi^{-1}(Z) \simeq X \setminus Z$
3. $\pi^{-1}(Z) \simeq Z \times \mathbb{P}(N_X Z)$

Let us show how this $\widehat{X}$ is defined in a local situation: X affine as Z and Z defined in local cordiantes $u_1, \ldots, u_n$ by $u_1 = \ldots = u_k = 0$ $(k > 1)$.

Blow-up

If $[U_1 : \ldots : U_k]$ are the projectives coordiantes on $\mathbb{P}^{k-1}$, the blow up $\widehat{X}$ is the subvariety of $X \times \mathbb{P}^{k-1}$

$$\widehat{X} \subset X \times \mathbb{P}^{k-1} \qquad \text{defined by } \forall i,j \in \{1,\ldots,k\} \quad u_i U_j = u_j U_i$$

If we are on the affine subset of $\mathbb{P}^{k-1}$ defined by $U_k = 1$, then local coordinates on $\widehat{X}$ are given by $U_1, \ldots, U_{k-1}, u_k, u_{k+1}, \ldots u_n$.and

$$\pi : (U_1, \ldots, U_{k-1}, u_k, u_{k+1}, \ldots u_n) \longmapsto (u_k U_1, \ldots, u_k U_{k-1}, u_k, u_{k+1}, \ldots u_n)$$

The projection $\overline{\mathcal{M}_{0,n+3}} \to (\mathbb{P}^1)^n$

The projection $p : \overline{\mathcal{M}_{0,n+3}} \to (\mathbb{P}^1)^n$ is an extesnion of the natural projection $\mathcal{M}_{0,n+3} \to (\mathbb{P}^1)^n$ which send $(0, z_1, ..., z_n, 1\infty)$ on $(z_1, ..., z_n)$.

Thinkink of erasing the symbol 0, 1 and ∞ :

$$(0, z_1, ..., z_n, 1\infty) \mapsto (z_1, ..., z_n)$$

is simple but useless to describe what happen on the boundary

question

In the Case $n = 3$ what is the image of the component given by $0z_1z_3|z_21\infty$?

- A geodesic surounding $0, z_1$ and z_3 have a lengh that tend to 0 when it tends to the boundary.
- Symbolically we have $0 = z_1 = z_3$ which is the equation of a line in $(\mathbb{P}^1)^3$. The component $0z_1z_3|z_21$ maps to that line ...

Description

In order to obtain a descrition of the image of the boundary component, we say that the points in the same subset of the partition are equal. More preciesely

- Components of types $s_i s_j|...$, $s_i \varepsilon|...$ with $\varepsilon \in \{0, 1, \infty\}$, give hyperplane $x_i = x_j$ and $x_i = 0, 1, \infty$;
- Partition of 3 points$|...$ (with at most one being $0, 1, \infty$) give codimension 2 afine space ;
- ...
- Partitions of types $\varepsilon z_1 \dots z_n|ab$ (with $\varepsilon = 0, 1, \infty$)give the points $(0, ..., 0)$ $(1, ..., 1)$ and $(\infty, ..., \infty)$.

Forgetful maps

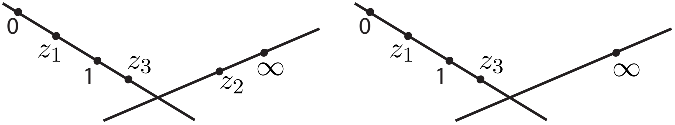
Let $n \geqslant 3$ and S a finte set we write $\overline{\mathcal{M}_{0,n}}S$ $\overline{\mathcal{M}_{0,n}}|S|$. Let S' be a subset of S with $|S'| \geqslant 3$. Then we have a canonical morphism (forgetfull map) $\Phi_T : \overline{\mathcal{M}_{0,S}} \to \overline{\mathcal{M}_{0,S'}}$ (with $T = S \setminus S'$) which delete the point indexed by elements of T and "smooth" the unstable component.

Example with $\mathcal{M}_{0,5}$

- The point $(0, 0)$.
- $0z_1z_2|1\infty \mapsto \underbrace{0z_1|1\infty}_{t_1=0} \times \underbrace{0z_2|1\infty}_{t_2=0}$

Some pictures

In $\overline{\mathcal{M}_{0,6}}$, $S = \{0, z_1, z_2, z_3, 1, \infty\}$, $T = \{z_2\}$ lets have a look to the component defined by $z_2\infty|0z_1z_31$:



Stable curve.

delete z_2

unstable curve

Figure:

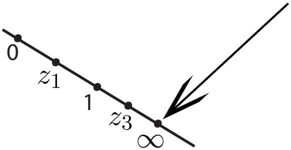


Figure: The “smooththing” or “contracting” is done in putting the last label at the node place

Crash down divisor for $\mathcal{M}_{0,5}$ and $\mathcal{M}_{0,6}$

$\overline{\mathcal{M}_{0,5}}$

- $\mathbf{01|z_1z_2\infty}$
- $\mathbf{1\infty|0z_1z_2}$
- $\mathbf{0\infty|z_1z_21}$

example in $\overline{\mathcal{M}_{0,6}}$

- $01|z_1z_2z_3\infty \mapsto$ point
- $1\infty|0z_1z_2z_3 \mapsto$ point
- $0\infty|z_1z_2z_31 \mapsto$ point
- $01z_3|z_1z_2\infty \mapsto$ line
- $z_1z_2z_3|01\infty \mapsto$ line
- ...