Double shuffle and moduli spaces of curves in genus 0

Introduction

For a p-tuple $\mathbf{k} = (k_1, \ldots, k_p)$ of positive integers and $k_1 \geqslant 2$, the multiple zeta value $\zeta(\mathbf{k})$ is defined as

$$\zeta(\mathbf{k}) = \sum_{n_1 > \ldots > n_p > 0} \frac{1}{n_1^{k_1} \cdots n_p^{k_p}}.$$

These values satisfy two families of algebraic (quadratic) relations known as double shuffle, or shuffle and stuffle described below.

- The stuffle comes from the above representation,
- the shuffle comes from the integral representation of the MZV.

Combinatorial stuffle

The stuffle product of a p-tuple $\mathbf{k} = (\mathbf{k}_0, k_p)$ ($\mathbf{k}_0 = (k_1, \ldots, k_{p-1})$) and a q-tuple $\mathbf{l} = (\mathbf{l}_0, l_q)$ ($\mathbf{l}_0 = (l_1, \ldots, l_{q-1})$) is defined recursively by the formula:

$$(\mathbf{k}) * (\mathbf{l}) = (\mathbf{k} * \mathbf{l}_0) \cdot l_q + (\mathbf{k}_0 * \mathbf{l}) \cdot k_p + (\mathbf{k}_0 * \mathbf{l}_0) \cdot (k_p + l_q) \qquad (1)$$

and $\mathbf{k} * () = () * \mathbf{k} = \mathbf{k}$.

If σ is a term of the formal sum $\mathbf{k} * \mathbf{l}$, we will write $\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})$.

Example

$$(n) * (m) = (n, m) + (m, n) + (n + m)$$

$$(u) * (v, w) = (u, v, w,) + (v, u, w) + (v, w, u) + (u + v, w) + (v, u + w)$$

Stuffle and multiple zeta values

Proposition (Stuffle relations)

Let $\mathbf{k} = (k_1, \ldots, k_p)$ *and* $\mathbf{l} = (l_1, \ldots, l_q)$ *be as above with* $k_1,\ l_1 \geqslant 2$. *Then we have:*

$$\zeta(\mathbf{k})\zeta(\mathbf{l}) = \left(\sum_{n_1 > \ldots > n_p > 0} \frac{1}{n_1^{k_1} \cdots n_p^{k_p}} \right) \left(\sum_{m_1 > \ldots > m_q > 0} \frac{1}{m_1^{l_1} \cdots m_q^{l_q}} \right) = \sum_{\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})} \zeta(\sigma).$$

Stuffle and multiple zeta values

Example

$$\zeta(k)\zeta(l) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{1}{n^k m^l}$$
$$= \sum_{n>m>0}\frac{1}{n^k m^l} + \sum_{m>n>0}\frac{1}{m^l n^k} + \sum_{n=m}\frac{1}{n^{k+l}}$$
$$= \zeta(k,l) + \zeta(l,k) + \zeta(k+l).$$

General case.

We split the summation domain of the product $\zeta(\mathbf{k})\zeta(\mathbf{l})$

$$\{0 < n_1 < \ldots < n_p\} \times \{0 < m_1 < \ldots < m_q\}$$

into all the domains that preserve the respective order of the n_i and the m_j and the boundary domains where some n_i are equal to some m_j. □

Combinatorial shuffle

Let $(e_1, \overline{e})$ and $(f_1, \overline{f})$ be respectively an n-tuple and an m-tuple of symbols. The shuffle product $(e_1, \overline{e}) ⧢ (f_1, \overline{f})$ is defined recursively by:

$$(e_1, \overline{e}) ⧢ (f_1, \overline{f}) = e_1 \cdot (\overline{e} ⧢ (f_1, \overline{f})) + f_1 \cdot ((e_1, \overline{e}) ⧢ \overline{f}) \qquad (2)$$

and $\overline{e} ⧢ () = () ⧢ \overline{e} = \overline{e}$.
Let $\overline{e}$ and $\overline{f}$ be an n-tuple and an m-tuple, if σ is a term of the formal sum $\overline{e} ⧢ \overline{f}$, we will write $\sigma \in \overline{k} ⧢ \overline{l}$.
Such an element is defined by a permutation of $[\![1, n+m]\!]$ that we shall denote by $\sigma \in \mathrm{sh}(\overline{k}, \overline{l})$.

Example

$XY ⧢ AB = XYAB + XAYB + XABY + AXYB + AXBY + ABXY$

Integral representation of the MZV

To the tuple $\mathbf{k}$, with $n = k_1 + \cdots + k_p$, we associate the n-tuple

$$\overline{k} = (\underbrace{0,\ldots,0}_{k_1-1 \text{ times}}, 1, \ldots, \underbrace{0,\ldots,0}_{k_p-1 \text{ times}}, 1) = (\varepsilon_n, \ldots, \varepsilon_1)$$

and the differential form:

$$\omega_{\mathbf{k}} = \omega_{\overline{k}} = (-1)^p \frac{dt_1}{t_1 - \varepsilon_1} \wedge \cdots \wedge \frac{dt_n}{t_n - \varepsilon_n}.$$

Then, setting $\Delta_n = \{0 < t_1 < \ldots < t_n < 1\}$, direct integration yields:

$$\zeta(\mathbf{k}) = \int_{\Delta_n} \omega_{\mathbf{k}}.$$

Integral representation of the shuffle relations

Proposition (Shuffle relations)

Let $\mathbf{k} = (k_1, \ldots, k_p)$ *and* $\mathbf{l} = (l_1, \ldots, l_q)$ *with* $k_1,\ l_1 \geqslant 2$. *Then*

$$\int_{\Delta_n} \omega_{\overline{k}} \int_{\Delta_m} \omega_{\overline{l}} = \sum_{\sigma \in \mathrm{sh}(\overline{k}, \overline{l})} \int_{\Delta_{n+m}} \omega_{\sigma}. \qquad (3)$$

Proof.

Let $n = k_1 + ... + k_p$ and $m = l_1 + ... + l_q$. Then we have:

$$\int_{\Delta_n} \omega_{\overline{k}} \int_{\Delta_m} \omega_{\overline{l}} = \int_{\Delta} \frac{dt_1}{1-t_1} \cdots \frac{dt_n}{t_n} \frac{dt_{n+1}}{1-t_{n+1}} \cdots \frac{dt_{n+m}}{t_{n+m}}.$$

The set $\Delta = \{0 < t_1 < \ldots < t_n < 1\} \times \{0 < t_{n+1} < \ldots < t_{n+m} < 1\}$ can be up, to codimension 1 sets, split into a union of simplexes

$$\Delta = \coprod_{\sigma \in \mathrm{sh}([\![1,n]\!],[\![n+1,m]\!])} \Delta_{\sigma} \qquad \text{with } \Delta_{\sigma} = \{0 < t_{\sigma(1)} < t_{\sigma(2)} < ... < t_{\sigma(n+m)} < 1\}.$$

MZV and Integration over a square

Example of cubical representation of the MZVs

We have $\zeta(2) = \int_{\Delta_2} \frac{dt_2}{t_2} \frac{dt_1}{1-t_1}$. The change of variables $t_2 = x_1$ and $t_1 = x_1 x_2$ gives:

$$\zeta(2) = \int_{[0,1]^2} \frac{dx_1}{x_1} \frac{x_1\, dx_2}{1 - x_1 x_2} = \int_{[0,1]^2} \frac{dx_1\, dx_2}{1 - x_1 x_2}.$$

This change of variables is nothing but the blow up of the point $(0,0)$ in the projective plane, in n dimensions it corresponds to a sequence of blow up given by:

$$t_n = x_1,\ t_{n-1} = x_1 x_2,\ \ldots,\ t_1 = x_1 \ldots x_n. \tag{4}$$

Example of cubical representation of the MZVs

Using the former change of variable :

$$t_n = x_1,\ t_{n-1} = x_1 x_2,\ \dots,\ t_1 = x_1 \dots x_n.$$

for $n = 4$ we write the multiple zeta values as follows:

$$zeta(4) = \int_{[0,1]^4} \frac{d^4x}{1 - x_1x_2x_3x_4} \qquad \zeta(2,2) = \int_{[0,1]^4} \frac{x_1x_2d^4x}{(1 - x_1x_2)(1 - x_1x_2x_3x_4)}$$

$$\zeta(2)\zeta(2) = \int_{[0,1]^4} \frac{1}{1 - x_1x_2} \frac{1}{1 - x_3x_4} d^4x.$$

For any variables α and β we have the equality:

$$\frac{1}{(1-\alpha)(1-\beta)} = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} + \frac{\beta}{(1-\beta)(1-\beta\alpha)} + \frac{1}{1-\alpha\beta}. \qquad (5)$$

Example of cubical representation of the MZVs

Setting $\alpha = x_1x_2$ and $\beta = x_3x_4$ and applying (5), we recover the stuffle relation:

$$\zeta(2)\zeta(2) = \int_{[0,1]^4} \left(\frac{x_1x_2}{(1-x_1x_2)(1-x_1x_2x_3x_4)} + \frac{x_3x_4}{(1-x_3x_4)(1-x_3x_4x_1x_2)} + \frac{1}{1-x_1x_2x_3x_4} \right) d^4x \quad (6)$$

$$\zeta(2)\zeta(2) = \zeta(2,2) + \zeta(2,2) + \zeta(4).$$

General Case

Let $\mathbf{k} = (k_1, \ldots, k_p)$ be as above and $n = k_1 + \cdots + k_p$.
We define $f_{k_1,\ldots,k_p}$ to be the function of n variables on $[0,1]^n$ given by:

$$f_{k_1,\ldots,k_p}(x_1, \ldots, x_n) = \frac{1}{1 - x_1 \cdots x_{k_1}} \frac{x_1 \cdots x_{k_1}}{1 - x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_1+k_2}} \frac{x_1 \cdots x_{k_1+k_2}}{1 - x_1 \cdots x_{k_1+k_2+k_3}} \cdots \frac{x_1 \cdots x_{k_1+\ldots+k_{p-1}}}{1 - x_1 \cdots x_{k_1+\cdots+k_p}}. \quad (7)$$

Proposition

For all p-tuples of integers $(k_1, \ldots, k_p)$ with $k_1 \geqslant 2$, we have (with $n = k_1 + \cdots + k_p$):

$$\zeta(k_1, \ldots, k_p) = \int_{[0,1]^n} f_{k_1,\ldots,k_p}(x_1, \ldots, x_n) d^n x. \quad (8)$$

Notation ...

To derive the stuffle relations in general using integrals and the functions $f_{k_1,\dots,k_p}$, we will use the following notation. Let $\mathbf{k}$ be a sequence $(k_1,\dots,k_p)$, $n = k_1 + \cdots + k_p$. We have n variables $x_1,\dots,x_n$.

Notation

- $\mathbf{a} = (a_1,\dots,a_r)$, we will write $\prod \mathbf{a} = a_1 \cdots a_r$.
- $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{x}(\mathbf{k},1) = (x_1,\dots,x_{k_1})$ and

  $$\mathbf{x}(\mathbf{k},i) = (x_{k_1+\cdots+k_{i-1}+1},\dots,x_{k_1+\cdots+k_i}),$$

  so the $\mathbf{x}$ is the concatenation of sequences $\mathbf{x}(\mathbf{k},1)\cdots\mathbf{x}(\mathbf{k},p)$.
- $\mathbf{x}(\mathbf{k},\leqslant i) = (x_1,\dots,x_{k_1+\cdots+k_i})$. If $\mathbf{k} = (\mathbf{k_0},k_p)$, $\mathbf{x}_0 = \mathbf{x}(\mathbf{k},\leqslant p-1) = (x_1,\dots,x_{k_1+\cdots+k_{p-1}})$.
- If $\mathbf{l}$ is a q-tuple with $l_1 + \cdots + l_q = m$ and $\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})$ then : y_σ is the sequence where $\mathbf{x}(\mathbf{k},i)$ (resp. $\mathbf{x}'(\mathbf{l},j)$) is in the position of k_i (resp. l_j) in σ.
  Components of σ of the form $k_i + l_j$ give rise to sequence $\mathbf{x}(\mathbf{k},i)\mathbf{x}'(\mathbf{l},j)$

We remark that for each $\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})$, $\prod y_\sigma = \prod \mathbf{x} \prod \mathbf{x}'$.

Integral Stuffle

Remark

Let $(k_1, \ldots, k_p) = (\mathbf{k_0}, k_p)$ be a sequence of integers. Then:

$$f_{k_1,\ldots,k_p}(\mathbf{x}(\mathbf{k},1),\ldots,\mathbf{x}(\mathbf{k},p)) = f_{k_1,\ldots,k_{p-1}}(\mathbf{x}(\mathbf{k},\leqslant p-1))\frac{\prod \mathbf{x}(\mathbf{k},\leqslant p-1)}{1-\prod \mathbf{x}(\mathbf{k},\leqslant p)} \tag{9}$$

$$= f_{k_1,\ldots,k_{p-1}}(\mathbf{x}_0)\frac{\prod \mathbf{x}_0}{1-\prod \mathbf{x}}. \tag{10}$$

Proposition

Let $\mathbf{k} = (k_1, \ldots, k_p)$ *and* $\mathbf{l} = (l_1, \ldots, l_q)$ *be two sequences with* $n = k_1 + \cdots + k_p$ *and* $m = l_1 + \cdots + l_q$. *Then:*

$$f_{k_1,\ldots,k_p}(\mathbf{x}(\mathbf{k},1),\ldots,\mathbf{x}(\mathbf{k},p)) \cdot f_{l_1,\ldots,l_q}(\mathbf{x}'(\mathbf{l},1),\ldots,\mathbf{x}'(\mathbf{l},q)) = \sum_{\sigma\in \mathrm{st}(\mathbf{k},\mathbf{l})} f_\sigma(y_\sigma). \tag{11}$$

Proof

We proceed by induction on the depth of the sequence. The recursion formula for the stuffle is given in (1).
If $p = q = 1$**:** This is nothing but formula (5):

$$f_n(\mathbf{x}(\mathbf{k},1))f_m(\mathbf{x}'(\mathbf{l},1)) = \frac{1}{1-\prod \mathbf{x}(\mathbf{k},\leqslant 1)} \cdot \frac{1}{1-\prod \mathbf{x}'(\mathbf{l},\leqslant 1)} = \frac{1}{1-\prod \mathbf{x}} \cdot \frac{1}{1-\prod \mathbf{x}'}$$

$$f_n(\mathbf{x}(\mathbf{k},1))f_m(\mathbf{x}'(\mathbf{l},1)) \overset{(5)}{=} \frac{\prod \mathbf{x}}{(1-\prod \mathbf{x})(1-\prod \mathbf{x}\prod \mathbf{x}')} + \frac{\prod \mathbf{x}'}{(1-\prod \mathbf{x}')(1-\prod \mathbf{x}'\prod \mathbf{x})} + \frac{1}{1-\prod \mathbf{x}\prod \mathbf{x}'}. \quad (12)$$

Proof

Inductive step: Let $(k_1, \dots, k_p) = (\mathbf{k_0}, k_p)$ et $(l_1, \dots, l_q) = (\mathbf{l_0}, l_q)$ be two sequences.

$$f_{\mathbf{k_0},k_p}(\mathbf{x}_0, \mathbf{x}(\mathbf{k}, p)) f_{\mathbf{l_0},l_q}(\mathbf{x_0}', \mathbf{x}'(\mathbf{l}, q)) = f_{\mathbf{k_0}}(\mathbf{x}_0) \frac{\prod \mathbf{x}(\mathbf{k}, \leqslant p-1)}{1 - \prod \mathbf{x}(\mathbf{k}, \leqslant p)} f_{\mathbf{l_0}}(\mathbf{x_0}') \frac{\prod \mathbf{x}'(\mathbf{l}, \leqslant q-1)}{1 - \prod \mathbf{x}'(\mathbf{l}, \leqslant q)}$$

$$\overset{(5)}{=} f_{\mathbf{k_0}}(\mathbf{x}_0) f_{\mathbf{l_0}}(\mathbf{x_0}') \cdot \left(\prod \mathbf{x}(\mathbf{k}, \leqslant p-1) \prod \mathbf{x}'(\mathbf{l}, \leqslant q-1)\right)$$

$$\left(\frac{\prod \mathbf{x}(\mathbf{k}, \leqslant p)}{(1 - \prod \mathbf{x}(\mathbf{k}, \leqslant p))(1 - \prod \mathbf{x}(\mathbf{k}, \leqslant p) \prod \mathbf{x}'(\mathbf{l}, \leqslant q))} \right.$$

$$+ \frac{\prod \mathbf{x}'(\mathbf{l}, \leqslant q)}{(1 - \prod \mathbf{x}'(\mathbf{l}, \leqslant q))(1 - \prod \mathbf{x}'(\mathbf{l}, \leqslant q) \prod \mathbf{x}(\mathbf{k}, \leqslant p))}$$

$$\left. + \frac{1}{(1 - \prod \mathbf{x}(\mathbf{k}, \leqslant p) \prod \mathbf{x}'(\mathbf{l}, \leqslant q))} \right).$$

Proof

Expanding and using the remark we obtain:

$$f_{\mathbf{k}_0,k_p}(\mathbf{x}_0,\mathbf{x}(\mathbf{k},p))f_{\mathbf{l}_0,l_q}(\mathbf{x}_0',\mathbf{x}'(\mathbf{l},q)) = \left(f_{\mathbf{k}_0,k_p}(\mathbf{x})f_{\mathbf{l}_0}(\mathbf{x}_0')\right)\cdot\frac{\prod\mathbf{x}\prod\mathbf{x}'_0}{1-\prod\mathbf{x}\prod\mathbf{x}'}$$

$$+\left(f_{\mathbf{k}_0}(\mathbf{x}_0)f_{\mathbf{l}_0,l_q}(\mathbf{x}')\right)\cdot\frac{\prod\mathbf{x}'\prod\mathbf{x}_0}{1-\prod\mathbf{x}'\prod\mathbf{x}}+\left(f_{\mathbf{k}_0}(\mathbf{x}_0)f_{\mathbf{l}_0}(\mathbf{x}_0')\right)\cdot\frac{\prod\mathbf{x}_0\prod\mathbf{x}'_0}{1-\prod\mathbf{x}\prod\mathbf{x}'}.$$

Hence, the product of functions $f_{k_1,\dots,k_p}$ and $f_{l_1,\dots,l_q}$ satisfies a recursion formula identical to the formula (1) that defines the stuffle product. Using induction, the proposition follows.

Corollary (integral representation of the stuffle)

Integrating the statement of the previous proposition over the cube and permuting the variables in each term of the LHS, we obtain:

$$\zeta(\mathbf{k})\zeta(\mathbf{l}) = \int_{[0,1]^n} f_{\mathbf{k}}d^nx\int_{[0,1]^m} f_{\mathbf{l}}d^mx = \int_{[0,1]^{n+m}}\sum_{\sigma\in\mathrm{st}(\mathbf{k},\mathbf{l})} f_\sigma\, d^{n+m}x = \sum_{\sigma\in\mathrm{st}(\mathbf{k},\mathbf{l})}\zeta(\sigma).$$

Shuffle and moduli spaces of curves

- Let $\mathbf{k}$ and $\mathbf{l}$ be as in the previous section, let $n = k_1 + \cdots + k_p$ and $m = l_1 + \cdots + l_q$.
- We will identify a point of $\mathcal{M}_{0,j+3}$ with a sequence $(0, z_1, \ldots, z_j, 1, \infty)$, the $z_i \in \mathbb{P}^1$ all distinct and distinc from $0, 1$ and ∞.
- Φ_j is the open cell in $\mathcal{M}_{0,j+3}(\mathbb{R})$ which is mapped onto Δ_j, the standard simplex, by the map: $\mathcal{M}_{0,j+3} \to (\mathbb{P}^1)^j$

$$(0, z_1, \ldots, z_j, 1, \infty) \mapsto (z_1, \ldots, z_j).$$

- $\omega_{\mathbf{k}}$ is the Kontsevich form corresponding to the tuple $(k_1, \ldots, k_p)$

Then we have:

$$\zeta(k_1, \ldots, k_p) = \int_{\Phi_n} \omega_{\mathbf{k}}.$$

Proposition

Let β *be the map defined by:*

$$\begin{array}{ccc} \mathcal{M}_{0,n+m+3} & \xrightarrow{\beta} & \mathcal{M}_{0,n+3} \times \mathcal{M}_{0,m+3} \\ (0, z_1, \ldots, z_{n+m}, 1, \infty) & \longmapsto & (0, z_1, \ldots, z_n, 1, \infty) \times (0, z_{n+1}, \ldots, z_{n+m}, 1, \infty). \end{array}$$

Then, if we write t_i *for the coordinate such that* $t_i(0, z_1, \ldots, z_{n+m}, 1, \infty) = z_i$*, we have:*

$$\beta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}) = \frac{dt_1}{1-t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \wedge \frac{dt_{n+1}}{1-t_{n+1}} \wedge \cdots \wedge \frac{dt_{n+m}}{t_{n+m}}.$$

Furthermore, if for $\sigma \in \mathrm{sh}((1, \ldots, n), (n+1, \ldots, n+m))$ *we write* Φ^{σ}_{n+m} *or* Φ_σ *for the open cell of* $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ *in which the points are in the same order as their indices are in* σ*, we have:*

$$\beta^{-1}(\Phi_n \times \Phi_m) = \coprod_{\sigma \in \mathrm{sh}((1,\ldots,n),(n+1,\ldots,n+m))} \Phi^{\sigma}_{n+m}.$$

Proof.

The first part is obvious.
In order to show that $\beta^{-1}(\Phi_n \times \Phi_m) = \coprod \Phi^{\sigma}_{n+m}$ we have to remember that a cell in $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ is given by a cyclic order on the marked points. Let $X = (0, z_1, \ldots, z_{n+m}, 1, \infty)$ be a point in $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ such that $\beta(X) \in \Phi_n \times \Phi_m$. The values of the z_i have to be such that:

$$0 < z_1 < \ldots < z_n < 1 \; (< \infty) \quad \text{and} \quad 0 < z_{n+1} < \ldots < z_{n+m} < 1 \; (< \infty). \tag{13}$$

However there is no order condition relating z_1 to z_{n+1} for example. So, points on $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ which are in $\beta^{-1}(\Phi_n \times \Phi_m)$ are such that the z_i are compatible with (13). That is exactly $\coprod_{\sigma \in \mathrm{sh}((1,\ldots,n),(n+1,\ldots,n+m))} \Phi^{\sigma}_{n+m}$. □

Proposition

The shuffle product can be seen as the change of variables:

$$\int_{\Phi_n \times \Phi_m} \omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}} = \int_{\beta^{-1}(\Phi_n \times \Phi_m)} \beta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}).$$

Proof.

Using the previous proposition, the right hand side of this equality is equal to

$$\sum_{\sigma \in \mathrm{sh}((1,\ldots,n),(n+1,\ldots,n+m))} \int_{\Phi^{\sigma}_{n+m}} \frac{dt_1}{1-t_1} \wedge \cdots \wedge \frac{dt_{n+m}}{t_{n+m}}.$$

- Then we permute the variables and change their names in order to have an integral over Φ_{n+m} for each term.
- As the form $\frac{dt_{\sigma(1)}}{1-t_{\sigma(1)}} \wedge \cdots \wedge \frac{dt_{\sigma(n+m)}}{t_{\sigma(n+m)}}$ does not have any pole on the boundary of Φ^{σ}_{n+m} all the integrals are convergent.

□

Stuffle and Moduli space of curves

The cubical coordinates on $\overline{\mathcal{M}_{0,r}}$ are defined by $u_1 = t_r$ and $u_i = t_{r-i+1}/t_{r-i+2}$ for $i < r$. This cubical system is well adapted to express the stuffle relations on the moduli spaces of curves.

Proposition

Let δ be the map defined by:

$$\begin{array}{ccc} \mathcal{M}_{0,n+m+3} & \stackrel{\delta}{\longrightarrow} & \mathcal{M}_{0,n+3} \times \mathcal{M}_{0,m+3} \\ (0, z_1, \ldots, z_{n+m}, 1, \infty) & \longmapsto & (0, z_{m+1}, \ldots, z_{m+n}, 1, \infty) \times (0, z_1, \ldots, z_m, z_{m+1}, \infty) \end{array}$$

Then, setting $\omega_{\mathbf{k}} = f_{\mathbf{k}}(u_1, \ldots, u_n)d^n u$ and $\omega_{\mathbf{l}} = f_{\mathbf{l}}(u_{n+1}, \ldots, u_{n+m})d^m u$ where the $f_{\mathbf{k}}$ are as in the work on $\mathbb{R}^n$ and the u_i are the cubical coordinates on moduli spaces, we have:

$$\delta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}) = f_{k_1,\ldots,k_p}(u_1, \ldots, u_n) f_{l_1,\ldots,l_q}(u_{n+1}, \ldots, u_{n+m}) d^{n+m} u$$

and

$$\delta^{-1}(\Phi_n \times \Phi_m) = \Phi_{n+m}.$$

Proof.

To prove the second statement, let $X = (0, z_1, \dots, z_{n+m}, 1, \infty)$ such that $\delta(X) \in \Phi_n \times \Phi_m$. Then the values of the z_i are real and have to verify:

$$0 < z_1 < \dots < z_m < z_{m+1} \; (< \infty) \quad \text{and} \quad 0 < z_{m+1} < \dots < z_{n+m} < 1 \; (< \infty). \tag{14}$$

These conditions show that $0 < z_1 < \dots < z_m < z_{m+1} < \dots < 1 < \infty$, so $X \in \Phi_{n+m}$.

To prove the first statement, we claim that δ is expressed in cubical coordinates by:

$$(u_1, \dots, u_{n+m}) \longmapsto (u_1, \dots, u_n) \times (u_{n+1}, \dots, u_{n+m}).$$

It is obvious to see that on the left hand factor the coordinates are not changed. For the right hand factor we have to rewrite the expression of the right side in terms of the standard set of representative on $\mathcal{M}_{0,m+3}$. We have:

$$(0, z_1, \dots, z_m, z_{m+1}, \infty) = (0, z_1/z_{m+1}, \dots, z_m/z_{m+1}, 1, \infty) = (0, t_1, \dots, t_m, 1, \infty)$$

in simplicial coordinates. This point is given in cubical coordinates on $\mathcal{M}_{0,m+3}$ by:

$$(t_m, t_{m-1}/t_m, \dots, t_1/t_2) = (z_m/z_{m+1}, \dots, z_1/z_2) = (u_{n+1}, \dots, u_{n+m}).$$

As a consequence of this discussion and the results of section 3, we have the following proposition.

Proposition

Using the Cartier decomposition (11), the stuffle product can be viewed as the change of variables:

$$\int_{\Phi_n \times \Phi_m} \omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}} = \int_{\delta^{-1}(\Phi_n \times \Phi_m)} \delta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}).$$

Remark

We should point out here the fact that the Cartier decomposition "does not lie in the moduli spaces of curves":
Forms appear in the decomposition which are not holomorphic on the moduli space.

Remark

- For example, in the Cartier decomposition of $f_{2,1}(u_1, u_2, u_3) f_{2,1}(u_4, u_5, u_6)$, there appears the term:

$$\frac{u_1 u_2 u_4 u_5 \, du_1 \, du_2 \, du_3 \, du_4 \, du_5 \, du_6}{(1 - u_1 u_2 u_4 u_5)(1 - u_1 u_2 u_3 u_4 u_5 u_6)}$$

  which is not a holomorphic differential form on $\mathcal{M}_{0,6}$.
- However, it is a well-defined convergent form on the standard cell where it is integrated. Changing the numbering of the variables (which is allowed because the standard cell is symmetric under the permutation of the variable) gives the equality with $\zeta(4, 2)$.
- This example represents the situation in the general case: when simply dealing with integrals, the non-holomorphic forms are not a problem. However, in the context of framed motives they are.